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Tides in Oceans Bounded by Meridians. IV. Series Solutions in Terms of Angular Width of Ocean: Semidiurnal Tides in Narrow Oceans. V. Solutions by Use of Finite Differences: Semidiurnal Tides

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TIDES IN OCEANS BOUNDED BY MERIDIANS

IV. SERIES SOLUTIONS IN TERMS OF ANGULAR WIDTH OF OCEAN: SEMIDIURNAL TIDES IN NARROW OCEANS

V. SOLUTIONS BY USE OF FINITE DIFFERENCES: SEMIDIURNAL TIDES

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PART IV. SERIES SOLUTIONS IN TERMS OF ANGULAR WIDTH OF OCEAN: SEMIDIURNAL TIDES IN NARROW OCEANS

1. Introduction

The first three parts of this series of memoirs (Proudman 1936; Doodson 1936, 1938) have been concerned with the tides in an ocean bounded by a complete meridian, and the main issue was that of the variation of the diurnal and semidiurnal tides with the variation in depth of the ocean. The methods used in the first three parts are derived from a general method due to Proudman, which will in due course be applied to other oceans.

Other methods of attacking the problems have been exploited. One line of investigation is to consider a very narrow ocean and to attempt to develop the solution, stage by stage, in terms of series of powers of the angular width of the ocean. The method has one outstanding feature in that it does not require the solution of a number of simultaneous equations, but it suffers from the disadvantage that it is only applicable to narrow oceans. The solution is illustrated for the semidiurnal tide (K_2) in an ocean 30° wide.

Special consideration is given to the elucidation of the motion in very narrow oceans and to the results indicated by analytical methods for certain special depths.

2. Notation

We shall denote by

θ, χ the co-latitude and east longitude, respectively, of any point;

μ the value of $\cos\theta$;

η the value of $2-\sin^2\theta$;

α half the width of the ocean in angle of longitude, the bounding meridians being $\chi = \pm\alpha$;

ψ the value of χ/α;

h the depth of the ocean, supposed to be constant;

a the radius of the earth;

t the time;
g the acceleration of gravity;
u, v components of current in the directions respectively of increasing θ, χ;
ζ the elevation of the free surface of the ocean above the undisturbed level;
$\bar{\zeta}$ the "equilibrium form" of ζ corresponding to the disturbing forces;
ζ' the value of $\zeta-\bar{\zeta}$;
H the maximum value of $\bar{\zeta}$ over the whole earth for the semidiurnal tide;
R the amplitude of tide at any place;
σ the speed of the harmonic constituent to be considered, where $\sigma = 2\pi/\text{period}$;
γ the lag of the phase of tide behind the phase of the equilibrium tide on the central meridian;
ω the angular rotation of the earth;
β the value of σ^2a^2/gh.

Further, we shall write $\zeta = \zeta_1 \cos \sigma t + \zeta_2 \sin \sigma t$, with a corresponding notation for u, v, $\bar{\zeta}$, ζ', where ζ_1, ζ_2 and analogous functions do not involve the time.

We shall also take $\sigma = 2\omega$, so that the harmonic constituent investigated is K_2.

The value of $\bar{\zeta}$ appropriate to a semidiurnal tidal constituent is given by

$$\bar{\zeta} = H \sin^2\theta \cos(\sigma t + 2\chi), \tag{2·1}$$

whence

$$\bar{\zeta}_1 = H \sin^2\theta \cos 2\chi, \quad \bar{\zeta}_2 = -H \sin^2\theta \sin 2\chi. \tag{2·2}$$

3. Fundamental equations

The fundamental differential equations are

$$\frac{\partial u}{\partial t} - 2\omega v \cos\theta = -\frac{g}{a}\frac{\partial \zeta'}{\partial \theta}, \tag{3·1}$$

$$\frac{\partial v}{\partial t} + 2\omega u \cos\theta = -\frac{g}{a\sin\theta}\frac{\partial \zeta'}{\partial \chi}, \tag{3·2}$$

$$\frac{\partial \zeta}{\partial t} + \frac{1}{a\sin\theta}\left\{\frac{\partial}{\partial\theta}(hu\sin\theta) + \frac{\partial}{\partial\chi}(hv)\right\} = 0, \tag{3·3}$$

and from these we derive

$$\frac{\sigma a}{g} u_1 \sin^3\theta = \sin\theta \frac{\partial \zeta_2'}{\partial \theta} + \cos\theta \frac{\partial \zeta_1'}{\partial \chi}, \tag{3·41}$$

$$\frac{\sigma a}{g} u_2 \sin^3\theta = -\sin\theta \frac{\partial \zeta_1'}{\partial \theta} + \cos\theta \frac{\partial \zeta_2'}{\partial \chi}, \tag{3·42}$$

$$\frac{\sigma a}{g} v_1 \sin^3\theta = \frac{\partial \zeta_2'}{\partial \chi} - \sin\theta\cos\theta \frac{\partial \zeta_1'}{\partial \theta}, \tag{3·51}$$

$$-\frac{\sigma a}{g} v_2 \sin^3\theta = \frac{\partial \zeta_1'}{\partial \chi} + \sin\theta\cos\theta \frac{\partial \zeta_2'}{\partial \theta}. \tag{3·52}$$

On eliminating the components of current from (3·3) we obtain

$$\nabla^2\zeta_1'+\frac{\partial^2\zeta_1'}{\partial\chi^2}+\eta\frac{\partial\zeta_2'}{\partial\chi}=-\beta H\sin^6\theta\cos 2\chi, \tag{3·61}$$

$$\nabla^2\zeta_2'+\frac{\partial^2\zeta_2'}{\partial\chi^2}-\eta\frac{\partial\zeta_1'}{\partial\chi}=\beta H\sin^6\theta\sin 2\chi, \tag{3·62}$$

where ∇^2 denotes a differential operator such that, if applied to any function Z, it yields

$$\nabla^2 Z=\left\{\sin^3\theta\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right)+\beta\sin^4\theta\right\}Z \tag{3·71}$$

$$=Z''\sin^2\theta-Z'\sin\theta\cos\theta+\beta Z\sin^4\theta \tag{3·72}$$

if dashes denote differentiation with respect to θ. The operator can also be written as

$$\nabla^2 Z=(1-\mu^2)^2\left(\frac{\partial^2 Z}{\partial\mu^2}+\beta Z\right). \tag{3·73}$$

4. Angular width of ocean as parameter

Write

$$\zeta_1'=Z_0+\alpha^2Z_2+\alpha^4Z_4+\ldots, \tag{4·11}$$

$$\zeta_2'=\alpha Z_1+\alpha^3Z_3+\ldots, \tag{4·12}$$

where Z_r, the coefficient of α^r, is a function of θ and ψ $(=\chi/\alpha.)$

Suppose that it is possible to satisfy independently for each power of α the differential equations (3·61) and (3·62) and the boundary conditions $v_1=v_2=0$. This requires $\bar{\zeta}_1$ and $\bar{\zeta}_2$ to be expanded in the form

$$\bar{\zeta}_1=H\sin^2\theta\cos 2\chi=H\sin^2\theta\sum_r C_r\frac{1}{r!}\psi^r\alpha^r \quad (r \text{ even}), \tag{4·21}$$

$$\bar{\zeta}_2=-H\sin^2\theta\sin 2\chi=-H\sin^2\theta\sum_r S_r\frac{1}{r!}\psi^r\alpha^r \quad (r \text{ odd}), \tag{4·22}$$

where

$$C_0=1,\quad C_2=-2^2,\quad C_4=2^4,\ldots, \tag{4·31}$$

$$S_1=2,\quad S_3=-2^3,\quad S_5=2^5,\ldots. \tag{4·32}$$

Then, on equating powers of α, we have

$$\frac{\partial^2 Z_{r+2}}{\partial\psi^2}+\eta\frac{\partial Z_{r+1}}{\partial\psi}+\nabla^2 Z_r=-H\beta\sin^6\theta\frac{C_r}{r!}\psi^r \quad (r \text{ even}), \tag{4·41}$$

$$\frac{\partial^2 Z_{r+2}}{\partial\psi^2}-\eta\frac{\partial Z_{r+1}}{\partial\psi}+\nabla^2 Z_r=H\beta\sin^6\theta\frac{S_r}{r!}\psi^r \quad (r \text{ odd}), \tag{4·42}$$

with the boundary conditions (3·51) and (3·52) yielding on $\psi^2 = 1$

$$\left[\frac{\partial Z_{r+1}}{\partial \psi} - \sin\theta\cos\theta\frac{\partial Z_r}{\partial \theta}\right] = 0 \quad (r \text{ even}), \tag{4·51}$$

$$\left[\frac{\partial Z_{r+1}}{\partial \psi} + \sin\theta\cos\theta\frac{\partial Z_r}{\partial \theta}\right] = 0 \quad (r \text{ odd}). \tag{4·52}$$

The square brackets are used to indicate boundary conditions with $\psi = \pm 1$. It is clear that when Z_r, Z_{r+1} are known then Z_{r+2} can be determined by ordinary processes of differentiation and integration in θ and ψ.

We shall omit the comma between r, s in the suffixes in the numerical cases, unless r or s is equal to or greater than 10.

It is known that ζ_1' must be a symmetrical function of ψ and that ζ_2' must be asymmetrical in ψ, and therefore we clearly have

$$\frac{\partial^2 Z_0}{\partial \psi^2} = 0, \quad \frac{\partial^2 Z_1}{\partial \psi^2} = 0, \quad \left[\frac{\partial Z_0}{\partial \psi}\right] = 0, \tag{4·61}$$

whence we obtain, if Z_{00}, Z_{11}, are functions only of θ,

$$Z_0 = Z_{00}, \tag{4·62}$$

$$Z_1 = Z_{11}\psi, \tag{4·63}$$

and it also follows that Z_2 is of the form $Z_{20} + \frac{1}{2!}Z_{22}\psi^2$, and in general Z_r is a polynomial of order r in ψ, with coefficients which are functions only of θ.

5. The recurrence equations

We can thus write

$$\zeta_1' = Z_{00} + \alpha^2(Z_{20} + \frac{1}{2!}\psi^2 Z_{22}) + \alpha^4(Z_{40} + \frac{1}{2!}\psi^2 Z_{42} + \frac{1}{4!}\psi^2 Z_{44}) + \ldots, \tag{5·11}$$

$$\zeta_2' = \alpha(\psi Z_{11}) + \alpha^3(\psi Z_{31} + \frac{1}{3!}\psi^3 Z_{33}) + \ldots, \tag{5·12}$$

where $Z_{r,s}$ is a function only of θ.

On substituting in (3·61) and (3·62), or on developing (4·41) and (4·42) and equating coefficients of α and also of ψ, we obtain the general recurrence equations

$$Z_{r+2,\,s+2} + \eta Z_{r+1,\,s+1} + \nabla^2 Z_{r,\,s} = 0 \quad (r \neq s),\ (r,\ s \text{ even}), \tag{5·21}$$

$$= -H\beta\, C_r \sin^6\theta \quad (r = s),\ (r,\ s \text{ even}), \tag{5·22}$$

$$Z_{r+2,\,s+2} - \eta Z_{r+1,\,s+1} + \nabla^2 Z_{r,\,s} = 0 \quad (r \neq s),\ (r,\ s \text{ odd}), \tag{5·31}$$

$$= H\beta\, S_r \sin^6\theta \quad (r = s),\ (r,\ s \text{ odd}). \tag{5·32}$$

It is evident that when Z_{00} and Z_{11} are determined, then straightforward processes of differentiation will yield $Z_{22}, Z_{33}, \ldots$, and similarly the sequence of values $Z_{20}, Z_{31}, Z_{42}, Z_{53}, \ldots$ all follow from a knowledge of the first two functions of the series, and so on for the sequences beginning with $Z_{r,0}$ and $Z_{r+1,1}$.

6. The boundary equations

On substituting the expressions for ζ_1' and ζ_2' in the right-hand side of (3·51) and (3·52), equating to zero, writing $\psi = 1$, and collecting coefficients of α, we obtain

$$Z_{r+1,1} = \sin\theta\cos\theta\frac{\partial}{\partial\theta}Z_{r,0} - P_r \quad (r \text{ even}), \tag{6·11}$$

$$Z_{r+2,2} = -\sin\theta\cos\theta\frac{\partial}{\partial\theta}Z_{r+1,1} - Q_r \quad (r \text{ odd}), \tag{6·12}$$

where

$$P_r = \sum_{s=2}^{r}\left(\frac{1}{s!}Z_{r+1,s+1}\right) - \sin\theta\cos\theta\frac{\partial}{\partial\theta}\sum_{s=2}^{r}\left(\frac{1}{s!}Z_{r,s}\right), \tag{6·21}$$

$$Q_r = \sum_{s=3}^{r+1}\left(\frac{1}{s!}Z_{r+2,s+1}\right) + \sin\theta\cos\theta\frac{\partial}{\partial\theta}\sum_{s=3}^{r+1}\left(\frac{1}{s!}Z_{r+1,s}\right). \tag{6·22}$$

We have seen that the sequences of functions

$$\begin{gathered} Z_{00}, Z_{11}, Z_{22}, Z_{33}, \ldots, \\ Z_{20}, Z_{31}, Z_{42}, Z_{53}, \ldots, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots, \end{gathered}$$

depend only on the first two functions of each sequence, and it is easily verified that the functions appearing in P_r and Q_r belong to the sequences associated with $Z_{r-2,0}$, $Z_{r-4,0}, \ldots, Z_{00}$, whereas the other functions $Z_{r,0}, Z_{r+1,1}, Z_{r+2,2}$ in (6·11) and (6·12) belong to the sequence associated with $Z_{r,0}$. If we suppose that the functions of the lower orders of sequence are first derived then it is evident that P_r and Q_r can be considered "known".

7. The differential equation for $Z_{r,0}$

It is clear from the preceding paragraph that there are three equations connecting the functions $Z_{r,0}, Z_{r+1,1}, Z_{r+2,2}$, namely (6·11), (6·12) and (6·22), with $s = 0$ in the latter equation, and these suffice to determine these three initial functions of the sequence associated with $Z_{r,0}$.

As an example, take $r = 0$, whence the equations become

$$Z_{22} + \eta Z_{11} + \nabla^2 Z_{00} = -H\beta\sin^6\theta, \tag{7·11}$$

$$Z_{11} = \sin\theta\cos\theta\frac{\partial}{\partial\theta}Z_{00}, \tag{7·12}$$

$$Z_{22} = -\sin\theta\cos\theta\frac{\partial}{\partial\theta}Z_{11}. \tag{7·13}$$

On eliminating Z_{22} and Z_{11} we obtain

$$\sin^3\theta(Z_0''\sin\theta+Z_0'\cos\theta+\beta Z_{00}\sin\theta) = -H\beta\sin^6\theta, \tag{7·14}$$

where dashes denote differentiation with respect to θ. This equation yields a finite solution

$$Z_{00} = \frac{H}{\beta-6}(4-\beta\sin^2\theta). \tag{7·15}$$

From this value of Z_{00} the values of Z_{11} and Z_{22} are obtainable quite simply from (7·12) and (7·13) and a cross check on Z_{22} is again obtained from (7·11).

Similar processes yield the general equation

$$Z_{r,0}''\sin\theta+Z_{r,0}'\cos\theta+\beta Z_{r,0}\sin\theta = \left\{-\sin\theta\cos\theta\frac{\partial}{\partial\theta}P_r+\eta P_r+Q_r\right\}\operatorname{cosec}^3\theta$$

$$= R_r\operatorname{cosec}^3\theta. \tag{7·2}$$

If we express all functions in powers of $\sin\theta$ it is readily verified that the right-hand side of this equation is a finite series in odd powers of $\sin\theta$. If we write the coefficient of $\sin^m\theta$ as A_m, and assume

$$Z_{r,0} = B_0+B_2\sin^2\theta+\dots, \tag{7·31}$$

we get the recurrence equation

$$B_m = \frac{(m+2)^2B_{m+2}-A_{m+1}}{m(m+1)-\beta}. \tag{7·32}$$

Since we have shown that after a certain value of m the coefficients A are zero, then we can write the higher values of B also zero, and so obtain the last coefficient B_m in terms of the last coefficient A_m, and thence work backwards until B_0 is uniquely determined, so long as $m(m+1)-\beta$ is never zero. We shall deal with the case $m(m+1) = \beta$ below.

8. The tides in a very narrow ocean

We have now shown that ζ_1' and ζ_2' can be expressed as in (5·11) and (5·12), that the coefficients of any power of α are finite series in powers of ψ, and that each of the coefficients in these latter series is a finite series in powers of $\sin\theta$. It is evident that the complete series would be of no value if they were not convergent for very small values of α, but there is no reason for supposing that such lack of convergence needs to be considered, so that we shall assume that when α is very small we can write

$$\zeta_1' = Z_{00}, \quad \zeta_2' = 0, \quad \cos 2\chi = 1,$$

and from (7·15) we obtain

$$\zeta_1 = \bar{\zeta}_1+\zeta_1' = H\sin^2\theta+\frac{H}{\beta-6}(4-\beta\sin^2\theta) = \frac{H}{\beta-6}(4-6\sin^2\theta), \tag{8·1}$$

which gives an amphidromic point when

$$\sin^2\theta = 0{\cdot}6667, \quad \theta = 54^\circ{\cdot}7. \tag{8·2}$$

If β becomes zero this gives the "corrected equilibrium tide" for a very narrow ocean.

It is remarkable that this solution for a very narrow ocean appears to give amphidromic systems around points which are independent of β. If, however, $\beta = m(m+1)$, where m is an integer, then (8·1) is only a Particular Integral, to which must be added a multiple of the Legendre Function $P_m(\cos\theta)$, the solution of

$$Z''_{00}\sin\theta + Z'_{00}\cos\theta + \beta Z_{00}\sin\theta = 0.$$

The questions arising from this are of a very important and general nature, and some consideration has been given to them in §§13 and 14. We may say at once, however, that we formed the conclusion that this series method is not suitable for the computation of these critical cases.

9. Considerations affecting choice of functions of latitude

It has been tacitly assumed that powers of $\sin\theta$ are convenient for the representation of the functions and we proceed to discuss the reasons for choosing harmonic functions in the form $\cos n\theta$ rather than powers of $\sin\theta$.

Since
$$\nabla^2 \sin^n\theta = n(n-2)\sin^2\theta - n(n-1)\sin^{n+2}\theta + \beta\sin^{n+4}\theta, \tag{9·1}$$

it is evident that the successive processes of differentiation involved in determining the functions in any sequence will quickly lead to very large numerical coefficients through the factors $n(n-2)$, $n(n-1)$. If, however, we choose the variable $\cos n\theta$ we have

$$\nabla^2\cos n\theta = \tfrac{1}{16}\beta\cos(n-4)\,\theta + \tfrac{1}{4}(n^2+n-\beta)\cos(n-2)\,\theta + \tfrac{1}{8}(-4n^2+3\beta)\cos n\theta + \tfrac{1}{4}(n^2-n-\beta)\cos(n+2)\,\theta + \tfrac{1}{16}\beta\cos(n+4)\,\theta, \tag{9·21}$$

with
$$\nabla^2\cos 2\theta = \tfrac{1}{4}(6-\beta) + \tfrac{1}{16}(-32+7\beta)\cos 2\theta + \tfrac{1}{4}(2-\beta)\cos 4\theta + \tfrac{1}{16}\beta\cos 6\theta, \tag{9·22}$$

$$\nabla^2(1) = \tfrac{3}{8}\beta - \tfrac{1}{2}\beta\cos 2\theta + \tfrac{1}{8}\beta\cos 4\theta. \tag{9·23}$$

Though more terms are involved in the processes, the numerical coefficients are much smaller than in the case where $\sin^n\theta$ is the variable. (The same result holds for many of the functions which are prominent in tidal theory, particularly the Legendre Functions, which involve very high coefficients if expressed in powers of $\cos\theta$, and very much smaller coefficients if expressed harmonically.) For these reasons we have chosen the harmonic representation, and consequently we write

$$\eta = 2 - \sin^2\theta = \tfrac{1}{2}(3+\cos 2\theta), \tag{9·3}$$

$$\sin\theta\cos\theta\frac{\partial}{\partial\theta}\cos n\theta = -\tfrac{1}{4}n\cos(n-2)\,\theta + \tfrac{1}{4}n\cos(n+2)\,\theta, \tag{9·4}$$

$$\eta\cos n\theta = \tfrac{1}{4}\cos(n-2)\,\theta + \tfrac{3}{2}\cos n\theta + \tfrac{1}{4}\cos(n+2)\,\theta, \tag{9·5}$$

$$\sin^6\theta = \tfrac{10}{32} - \tfrac{15}{32}\cos 2\theta + \tfrac{6}{32}\cos 4\theta - \tfrac{1}{32}\cos 6\theta. \tag{9·6}$$

10. The differential equation for $Z_{r,0}$ in terms of $\cos n\theta$

Special consideration needs to be given to the solution of the differential equation (7·2). The right-hand side of this equation is

$$R_r/\sin^3\theta,$$

and R_r will be represented by a finite series of harmonics. The processes of dividing such series by $\sin\theta$ have been considered in Part II (Doodson 1936, p. 298), and the continued application leads to the following results.

We can write

$$\left.\begin{aligned} R_r/\sin^3\theta &= (p_0+p_2\cos 2\theta+\ldots+p_m\cos m\theta)/\sin^3\theta \\ &= -2\{p_1\sin\theta+\ldots+p_{m-1}\sin(m-1)\theta\}/\sin^2\theta \\ &= -4\{p'_0+p'_2\cos 2\theta+\ldots+p'_{m-2}\cos(m-2)\theta\}/\sin\theta \\ &= 8\{p'_1\sin\theta+\ldots+p'_{m-3}\sin(m-3)\theta\}, \end{aligned}\right\} \qquad (10\cdot1)$$

provided that R_r is truly divisible by $\sin^3\theta$, which requires

$$\sum_0^m p_n = 0, \quad (n \text{ even}) \qquad (10\cdot21)$$

$$\sum_2^m (n-2)^2 p_n = 0 \quad (n \text{ even}). \qquad (10\cdot22)$$

The values of p'_r follow very simply from the general formula

$$p'_r = p_{r+3}+3p_{r+5}+6p_{r+7}+\ldots, \qquad (10\cdot31)$$

where the coefficients are $\quad 1,\ 3,\ 6,\ 10,\ 15,\ 21,\ 28,\ 36,\ldots, \qquad (10\cdot32)$

the third differences being zero. Thus we get, as examples

$$\begin{aligned} p'_{m-3} &= p_m, \\ p'_{m-5} &= p_{m-2}+3p_m, \\ p'_{m-7} &= p_{m-4}+3p_{m-2}+6p_m. \end{aligned}$$

The conditions (10·21) and (10·22) were tested and very small amendments made in the coefficients p so as to satisfy these relations. The sequence of numbers in (10·32) was written in a vertical column and placed alongside the values of p to facilitate the cross-multiplication.

The next procedure is to write

$$Z_{r,0} = a_0+a_2\cos 2\theta+\ldots,$$

whence substitution in the differential equation yields

$$a_n = \frac{-16p'_{n+1}+F_n a_{n+2}}{G_n}, \qquad (10\cdot41)$$

where $$F_n = (n+1)(n+2) - \beta, \tag{10·42}$$

$$G_n = n(n+1) - \beta, \quad G_0 = -2\beta. \tag{10·43}$$

The values of a_n are obtained by working backwards as explained in § 7.

11. Illustration for $\beta = 10$

The numerical processes involved are very much simpler than would appear at first sight, provided that the functions and appropriate multipliers are tabulated in a proper manner.

The coefficients arising from the operation $\nabla^2 \cos n\theta$ are obtained from (9·21), (9·22) and (9·23) as follows:

	Coefficients of				
n	$\cos(n-4)\theta$	$\cos(n-2)\theta$	$\cos n\theta$	$\cos(n+2)\theta$	$\cos(n+4)\theta$
0	—	—	3·75	−5	1·25
2	—	−1	2·375	−2	0·625
4	0·625	2·5	−4·25	0·5	0·625
6	0·625	8	−14·25	5	0·625
8	0·625	15·5	−28·25	11·5	0·625
10	0·625	25	−46·25	20·0	0·625

The table can readily be extended as far as is required, since the third differences are zero.

If $Z_{r,s}$ is expressed as $\Sigma a_n \cos n\theta$ and $Z_{r+1,s+1}$ as $\Sigma b_n \cos n\theta$ then it is clear that the coefficient of $\cos 6\theta$, for example, in $Z_{r+2,s+2}$, from (5·21), is

$$\begin{array}{ll} -0·625\,a_2 & \\ -0·5\quad a_4 & -0·25\,b_4 \\ 14·25\quad a_6 & -1·50\,b_6 \\ -15·5\quad a_8 & -0·25\,b_8 \\ -0·625\,a_{10} & \end{array}$$

It is a simple matter to lay out a compact table of such groups of multipliers, and to compute the required coefficients by laying the folded table of multipliers against the coefficients of the harmonic terms in the functions. A similar table but with reversed signs in the second column will be usable for equation (5·31). The same tables are used for (5·22) and (5·32) using (4·31), (4·32) and (9·6).

The initial functions Z_{00}, Z_{11}, Z_{22}, were obtained very readily from (7·15), (7·12) and (7·13), yielding

	Coefficients of $H \cos n\theta$		
n	Z_{00}	Z_{11}	Z_{22}
0	−0·25	−0·625	0·000
2	1·25	0·000	0·625
4	—	0·625	0·000
6	—	—	−0·625

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TABLE 1. VALUES OF $\frac{1}{s!} Z_{r,s}$, $\beta = 10$

Coefficients of cos $n\theta$

r	s	$n = 0$	2	4	6	8	10	12	14	16
0	0	−0·2500	1·2500							
1	1	−0·6250	0·0000	0·6250						
2	0	−0·5612	−3·0827	−0·2409	0·2181					
	2	0·0000	0·3125	0·0000	−0·3125					
3	1	−2·795	6·803	−4·837	0·697	0·132				
	3	1·393	−2·188	1·198	−0·313	−0·091				
4	0	−2·645	−5·760	−1·790	−1·504	0·374	0·055			
	2	0·832	−1·309	−1·426	2·643	−0·656	−0·085			
	4	0·161	−0·256	0·280	−0·391	0·184	0·022			
5	1	−4·508	10·405	1·562	−8·274	0·109	0·681	0·024		
	3	3·691	−3·713	−2·121	2·156	0·429	−0·422	−0·021		
	5	−0·581	0·734	0·003	−0·187	−0·051	0·078	0·004		
6	0	−1·395	0·261	7·157	−0·658	−5·887	0·617	0·531	0·014	
	2	4·718	−3·937	−3·448	−5·390	9·784	−0·933	−0·772	−0·020	
	4	−1·447	1·829	−1·645	3·415	−2·664	0·298	0·209	0·006	
	6	0·011	0·001	0·051	−0·225	0·232	−0·046	−0·023	−0·001	
7	1	−12·94	16·82	−28·92	59·04	−30·54	−9·29	5·45	0·39	
	3	6·42	−9·14	14·03	−23·38	8·77	6·04	−2·48	−0·27	
	5	−1·15	1·88	−2·16	2·20	0·05	−1·29	0·42	0·06	
	7	0·09	−0·14	0·10	−0·10	−0·02	0·10	−0·04	0·00	
8	0	−24·2	−53·3	−65·3	29·0	38·7	−34·1	−6·7	6·2	0·4
	2	7·9	−17·7	32·5	2·2	−74·9	48·7	10·5	−8·6	−0·5
	4	−3·5	5·5	−2·5	−11·3	25·2	−13·0	−2·7	2·1	0·1
	6	0·5	−0·6	−0·5	2·5	−3·3	1·5	0·3	−0·2	0·0
	8	0·0	0·0	0·0	−0·1	0·2	−0·1	0·0	0·0	0·0

Coefficients of cos $n\theta$

r	s	$n=0$	2	4	6	8	10	12	14	16	18	20	22
9	1	−3	72	107	−540	498	21	−227	64	8			
	3	16	−16	−73	218	−185	−33	101	−24	−4			
	5	−3	3	10	−26	17	11	−15	3	1			
	7	—	—	−1	1	0	−2	1					
10	0	109	290	562	−517	−256	689	−127	−261	91	10		
	2	33	9	−327	280	615	−950	105	373	−122	−14		
	4	−11	9	25	31	−223	246	−16	−91	28	3		
	6	1	−1	1	−13	33	−29	2	9	−3			
	8	—	—	—	1	−3	2	—	−1				
11	1	−193	−198	−997	5808	−7581	172	7450	−5563	904	192	7	
	3	20	−37	630	−2425	2970	223	−3227	2251	−309	−92	−4	
	5	−4	11	−90	302	−328	−100	449	−280	27	14	1	
	7	—	−1	6	−17	13	17	−35	18	−1	−1		
	9	—	—	—	—	—	−1	2	−1				
12	0	—	—	—	—	—	—	—	—	—	—	—	—
	2	−65	−290	3610	−5434	−6122	19840	−9561	−10436	10895	−2005	−416	−14
	4	2	38	−419	285	2346	−4672	1655	2757	−2530	441	93	3
	6	2	−4	11	66	−347	506	−142	−284	242	−40	−9	
	8	—	—	1	−10	28	−34	9	16	−13	2		
	10	—	—	—	1	−1	2	—	−1				

From Z_{00}, Z_{11} all the required functions with $r = s$ readily follow by the processes explained above.

The next sequence with $s = r-2$ was computed as follows:

n	P_2	Q_2	R_2 p_n	$R_2/\sin^3\theta$ p'_{n+1}	F_n	G_n	Z_{20} a_n
0	4·335937	1·738282	3·320312	0·839848	−8	−20	−0·561193
2	−6·562500	−2·220053	−6·184897	−0·800780	2	−4	−3·082678
4	2·968750	0·494792	4·947917	0·423177	20	10	−0·240885
6	−0·937500	−0·185547	−3·378905	−0·436198	46	32	0·218099
8	0·195313	0·266926	1·731771				
10	—	−0·094400	−0·436198				

The values of the coefficients of the harmonic terms in P and Q are very easily computed from (6·21) and (6·22) using (9·4), and R is the sum. The conditions indicated in (10·21) and (10·22) are both satisfied, and then the computation of p' follows by use of the multipliers in (10·32). Finally the values of a_n in the series Z_{20} follow from (10·41) with the values of F and G indicated in (10·42) and (10·43). From the value of Z_{20} the values of Z_{31} and Z_{42} follow from (6·11) and (6·12), again using (9·4), and the whole of the operations can be checked by computing Z_{42} from Z_{20} and Z_{11} according to the formula (5·21), after which the sequence can be continued by the use of the recurrence formulae (5·21) and (5·22).

Similarly, Z_{40}, Z_{60}, Z_{80}, $Z_{10,0}$ have been computed, and the resulting sequences tabulated as far as $r = 12$.

A valuable check to the operations involved in the recurrence formulae was obtained by computing the coefficient of $\sin^2\theta$ in each function, for it can be readily shown that these coefficients for any sequence

$$Z_{r,0},\quad Z_{r,2},\quad Z_{r,4},\quad Z_{r,6},\quad Z_{r,8},\quad Z_{r,10},\ldots,$$

are proportional to $\quad 1,\quad +2,\quad -2^2,\quad -2^3,\quad 2^4,\quad 2^5,\ldots.$

This afforded a very ready and sensitive check, because the coefficient of $\sin^2\theta$ is equal to

$$-\Sigma\tfrac{1}{2}n^2a_n, \tag{11·1}$$

where $Z = \Sigma a_n \cos n\theta$.

In all the calculations a very high order of accuracy, to six or more significant figures, was maintained, but the resulting functions have been tabulated in table 1 in abbreviated form, and the factor $1/s!$ has been incorporated as indicated in (5·11) and (5·12).

12. General remarks on the solution

In table 2 are given values of the functions Z_r as defined in (4·11), obtained by specifying $\psi = 0$, 0·333, 1·000 in the definitions of (5·11) and (5·12), and computed for specific values of θ with $\beta = 10$. The values of ζ'_1 and ζ'_2 are readily obtained on multiplying by the appropriate powers of α for any given value of α. It is evident that the rate of convergence is very slow so that the solution can only be exploited for small values of α.

Table 2. Values of functions Z_r for specified values of ψ

$\beta = 10$, coefficients of H

ψ	θ	$r = 0$	$r = 2$	$r = 4$	$r = 6$	$r = 8$	$r = 10$
—	0°	1·000	−3·667	−11·27	0·64	−109·3	590
0	20	0·708	−3·074	−7·02	5·07	−90·5	378
	40	−0·033	−0·979	−0·88	−12·06	14·6	556
	60	−0·875	1·319	−0·59	−2·60	51·8	−1082
	80	−1·425	2·042	2·22	3·00	−30·1	389
	90	−1·500	2·063	3·15	0·17	−4·9	170
0·333	20	0·708	−3·030	−7·14	4·61	−88·8	385
	40	−0·033	−0·956	−0·87	−10·19	9·8	502
	60	−0·875	1·267	−0·01	−2·80	54·9	−1015
	80	−1·425	2·027	2·18	4·07	−26·5	378
	90	−1·500	2·063	2·89	2·32	−9·9	232
1·000	20	0·708	−2·68	−8·05	0·98	−77·7	423
	40	−0·033	−0·77	−0·62	2·11	−17·0	132
	60	−0·875	0·85	4·28	−1·86	61·1	−481
	80	−1·425	1·90	2·59	6·94	1·2	222
	90	−1·500	2·06	1·90	10·18	−26·7	443
ψ	θ	$r = 1$	$r = 3$	$r = 5$	$r = 7$	$r = 9$	$r = 11$
0·333	20°	−0·172	0·375	2·35	0·1	−5	−88
	40	−0·404	0·894	0·33	−14·7	202	−1917
	60	−0·313	−0·989	−6·06	24·7	−367	6315
	80	−0·049	−4·239	−2·78	−28·4	167	−4324
	90	0·000	−4·815	−1·73	−42·7	245	−268
1·000	20	−0·516	1·272	6·57	0·8	9	−273
	40	−1·212	2·659	4·19	−30·8	435	−3902
	60	−0·938	−1·526	−11·54	50·3	−714	12364
	80	−0·146	−8·712	−5·51	−54·4	317	−8479
	90	0·000	−10·000	−2·92	−85·9	496	−584

It had been hoped that some indication might have been yielded by the values of the functions as to the way in which the functions tended to behave for larger values of r, but the hope was vain. If we restrict the contribution to $0{\cdot}01H$ for the larger values of r, we find that the solution is restricted to $\alpha < 0{\cdot}3$, and accordingly it has been illustrated for the value $\alpha = 0{\cdot}26418$, corresponding to an ocean 30° wide.

From the resulting values of ζ_1', ζ_2', the values of ζ_1 and ζ_2 have been obtained by adding respectively $\bar{\zeta}_1 = H\sin^2\theta\cos 2\chi$, $\bar{\zeta}_2 = -H\sin^2\theta\sin 2\chi$, and table 3 gives the values of ζ_1', ζ_2', ζ_1, ζ_2, R, γ where

$$\zeta = R\cos(\sigma t - \gamma) = \zeta_1\cos\sigma t + \zeta_2\sin\sigma t.$$

Figure 1 gives the cotidal lines for γ at intervals of 30°, and the co-range lines at intervals of one-fifth of the maximum amplitude.

An amphidromic point occurs on $\chi = 0$, $\theta = 58°$, which may be compared with the position given in (8·2); a difference of 3° exists between the positions of the amphidromic point in a very narrow ocean and in an ocean 30° wide.

TABLE 3. VALUES OF ζ_1', ζ_2', ζ_1, ζ_2, R, γ

$\beta = 10$, ocean 30° wide. Semidiurnal tide K_2

ψ	θ	ζ_1'	ζ_2'	ζ_1	ζ_2	R	γ
0	0°	0·694	—	0·694	—	0·69	0°
	20	0·465	—	0·582	—	0·58	0
	40	−0·107	—	0·306	—	0·31	180
	60	−0·789	—	−0·039	—	0·04	180
	80	−1·274	—	−0·304	—	0·30	180
	90	−1·344	—	−0·344	—	0·34	180
0·333	20	0·467	−0·036	0·582	−0·056	0·58	354
	40	−0·105	−0·090	0·301	−0·162	0·34	332
	60	−0·789	−0·105	−0·050	−0·235	0·24	258
	80	−1·274	−0·095	−0·319	−0·263	0·41	220
	90	−1·344	−0·091	−0·359	−0·264	0·45	216
1·000	20	0·485	−0·104	0·586	−0·162	0·61	345
	40	−0·088	−0·266	0·269	−0·473	0·54	300
	60	−0·797	−0·282	−0·147	−0·657	0·67	257
	80	−1·280	−0·208	−0·440	−0·692	0·82	238
	90	−1·346	−0·188	−0·480	−0·688	0·84	235

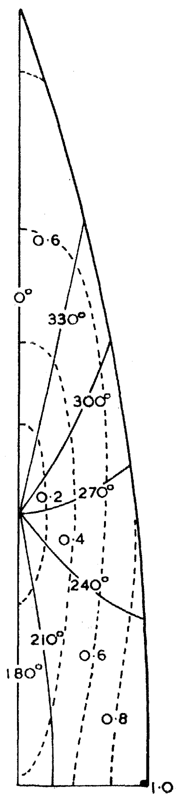

FIGURE 1. Semidiurnal tide in ocean 30° wide. $\beta = 10$, depth = 5·50 miles, factor = 0·84H.

13. APPLICATION OF METHOD TO NON-ROTATING OCEAN, $\beta = 20$

With a view to the discussion of the critical case where $\beta = m(m+1)$ and m is an integer, the simpler problems associated with a non-rotating earth have been considered, using the methods of this paper.

The equation to be solved is

$$Z_{r+2,\,s+2} + \vartheta^2 Z_{r,\,s} = 0 \qquad (r \neq s), \tag{13·11}$$

$$= -H\beta C_r \sin^4\theta \qquad (r = s,\ \text{even}), \tag{13·12}$$

$$= H\beta S_r \sin^4\theta \qquad (r = s,\ \text{odd}), \tag{13·13}$$

where

$$C_r = 2^0,\ -2^2,\ 2^4, \ldots \qquad (r\ \text{even}), \tag{13·14}$$

$$S_r = 2,\ -2^3,\ 2^5, \ldots \qquad (r\ \text{odd}), \tag{13·15}$$

and

$$\vartheta^2 Z = \sin\theta \frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial Z}{\partial\theta}\right) + \beta Z \sin^2\theta. \tag{13·16}$$

We shall take the same notation as in (5·11) and (5·12). The boundary conditions on $\chi = \alpha$ follow from

$$\frac{\partial \zeta_1'}{\partial \psi} = 0, \quad \frac{\partial \zeta_2'}{\partial \psi} = 0, \tag{13·17}$$

for each power of α, yielding

$$\left.\begin{aligned} &[Z_{22}] = 0, && [Z_1] = 0, \\ &\left[\frac{1}{3!}Z_{44}+Z_{42}\right] = 0, && \left[\frac{1}{2!}Z_{33}+Z_{31}\right] = 0, \\ &\dots && \dots \end{aligned}\right\} \tag{13·2}$$

From (13·12) and (13·13) we obtain

$$\left.\begin{aligned} &Z_{22}+\vartheta^2 Z_{00} = -20H\sin^4\theta, && Z_{33}+\vartheta^2 Z_{11} = 40H\sin^4\theta, \\ &Z_{44}+\vartheta^2 Z_{22} = 80H\sin^4\theta, && \dots \\ &Z_{42}+\vartheta^2 Z_{20} = 0, && \dots \end{aligned}\right\} \tag{13·3}$$

and we immediately deduce

$$\left.\begin{aligned} &Z_{22} = 0, && Z_{11} = 0, \\ &Z_{42} = -\tfrac{40}{3}H\sin^4\theta, && Z_{31} = -20H\sin^4\theta, \\ &Z_{44} = 80H\sin^4\theta, && Z_{33} = 40H\sin^4\theta. \end{aligned}\right\} \tag{13·4}$$

It may be noted that the sequence Z_{11}, Z_{22}, Z_{33}, Z_{44}, ... is independent of the value Z_{00} and similarly Z_{31}, Z_{42}, ... are independent of the value of Z_{20}. For these functions Z_{00} and Z_{20} we have

$$\vartheta^2 Z_{00} = -20H\sin^4\theta, \tag{13·51}$$

$$\vartheta^2 Z_{20} = \tfrac{40}{3}H\sin^4\theta, \tag{13·52}$$

from which we immediately deduce

$$Z_{20} = -\tfrac{2}{3}Z_{00}. \tag{13·53}$$

If we assume a series $\qquad Z_{00} = a_0 + a_2\sin^4\theta + \dots,$

then we obtain from (13·16) and (13·51)

$$a_0(20\sin^2\theta) + a_2(2^2\sin^2\theta - 14\sin^4\theta) + a_4(4^2\sin^4\theta - 0\sin^6\theta) + \dots = -20H\sin^4\theta. \tag{13·61}$$

It is evident that we can take $a_6 = a_8 = \dots = 0$, and that we then have two alternative solutions

$$(1)\quad a_0 = a_2 = 0 \text{ with } Z_{00} = -\tfrac{5}{4}H\sin^4\theta, \tag{13·62}$$

$$(2)\quad a_4 = 0 \text{ with } Z_{00} = \tfrac{2}{7}H(1-5\sin^2\theta). \tag{13·63}$$

The latter is in conformity with the general solution for a non-critical value of β as given in (7·15).

There is no criterion whatever for the ignoration of either of these solutions. We can take the general solution to be

$$Z_{00} = f\{\tfrac{2}{7}H(1-5\sin^2\theta)\} - g\{\tfrac{5}{4}H\sin^4\theta\}, \tag{13·64}$$

provided that

$$f+g = 1. \tag{13·65}$$

Hence f and g may both be very large, provided they satisfy the last equation, but one of them must then be negative, and the limiting case, for which there is resonance, is when f and g are opposite in sign and infinite in magnitude, when we get

$$Z_{00} \rightarrow P_4(\cos\theta) \times \infty. \tag{13·66}$$

The indeterminacy, of course, implies the existence of a free oscillation.

A solution for the case $\beta = 20$ was given by Proudman and Doodson (1927) as follows:

$$\zeta_1' = -\tfrac{5}{3} H \sin^4\theta \cos 2\chi + \tfrac{5}{6} H \frac{\sin 2\alpha}{\sin 4\alpha} \sin^4\theta \cos 4\chi, \tag{13·71}$$

$$\zeta_2' = \tfrac{5}{3} H \sin^4\theta \sin 2\chi - \tfrac{5}{6} H \frac{\cos 2\alpha}{\cos 4\alpha} \sin^4\theta \sin 4\chi, \tag{13·72}$$

and it is evident at once that this solution, having a factor $\sin^4\theta$, takes no account of the solution indicated by (13·63).

If we write

$$\left.\begin{aligned} \cos 2\chi &= \cos 2\alpha\psi = 1 - 2\alpha^2\psi^2 + \tfrac{2}{3}\alpha^4\psi^4 - \ldots, \\ \cos 4\chi &= \cos 4\alpha\psi = 1 - 8\alpha^2\psi^2 + \tfrac{32}{3}\alpha^4\psi^4 - \ldots, \\ \sec 2\alpha &= 1 + 2\alpha^2 + \tfrac{10}{3}\alpha^4 + \ldots, \\ \sec 4\alpha &= 1 + 8\alpha^2 + \tfrac{160}{3}\alpha^4 + \ldots, \end{aligned}\right\} \tag{13·73}$$

and expand these solutions in the forms discussed above, we very readily obtain the same functions as are given in (13·4), with the relations (13·53) and (13·62).

The investigation immediately above has an added interest, since it is known that the development of the solution (13·71) and (13·72) must be convergent for $\alpha < \pi/8$; thus to some extent the validity of the general processes is justified. If the developments of (13·4) are continued then the successive ratios of the coefficients of any sequence with the same value of s, such as $Z_{31}, Z_{51}, Z_{71}, \ldots$ very rapidly approach the ratio $(8/\pi)^2$, indicating that the coefficients ultimately become proportional to the coefficients of powers of α in

$$\left\{1 - \left(\frac{8\alpha}{\pi}\right)^2\right\}^{-1}.$$

The behaviour of the coefficients in these sequences thus readily indicates the values of α for which resonance occurs. Unfortunately, in the general case, no such simple indications were discovered.

14. Further considerations for the critical values of β

The preceding exposition reveals that the solution given in 1927 for the tides on a non-rotating earth was not complete. It is readily shown that the difference between the two solutions (13·62) and (13·63) is proportional to $P_4(\cos\theta)$ and that an arbitrary multiple of $P_4(\cos\theta)$ should be added to the expression for ζ_1' quoted in (13·71).

Proudman has considered the tides in a very narrow ocean by a different method and the following exposition is extracted from notes supplied by him.

If the ocean is very narrow then the transverse velocity v can be taken as zero, and the differential equations yield

$$\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\zeta}{\partial\theta}\right)+\beta\zeta=-6\bar{\zeta}, \tag{14·1}$$

provided that we write $\bar{\zeta}$ in the "corrected" form

$$\bar{\zeta}=H(\sin^2\theta-\tfrac{2}{3})\cos\sigma t. \tag{14·2}$$

The general solution of this differential equation is

$$\zeta=-\frac{6\bar{\zeta}}{\beta-6}+\{AP_m(\cos\theta)+BQ_m(\cos\theta)\}\cos\sigma t, \tag{14·3}$$

where
$$\beta=m(m+1), \tag{14·4}$$

and P, Q are Legendre Functions.

For finiteness from pole to pole, we must have m an integer, and since the semi-diurnal tides must be symmetrical about the equator, we must have m an even integer. Hence the solution becomes

$$\zeta=-\frac{6\bar{\zeta}}{\beta-6}+AP_m(\cos\theta)\cos\sigma t, \tag{14·5}$$

where m is an even integer, and A is arbitrary.

This expression reveals the distinction between the "free oscillations" which may become resonant, as for $\beta=6$, and the "free oscillations" which are non-resonant, as for $\beta=20$.

If m is not an even integer then $A=0$ and we get the same expression for ζ as has been obtained in previous paragraphs.

According to these investigations it would appear that any solutions for β *nearly equal* to 20 will be comparable with one another, but for β *exactly equal* to 20 we may get something very different. The former cases would give only one amphidromic point between the pole and the equator, while the latter critical case may give two amphidromic points.

In addition to $\beta=20$, one of the critical cases is that for $\beta=6$. According to the expression (14·5), resonance is indicated with $\beta=6$, though the theory does not indicate resonance for $\beta=20$. It should be noted that in a second attack on the problem of tides on a non-rotating earth, Proudman (1929, p. 211) discovered that for an ocean 180° wide, if the value of β is made to approach the value 6, a finite expression is obtained, so that resonance does not then occur.

15. Developments in terms of β

In order to determine the character of the solution near the critical points, the values of the functions have been computed in terms of β.

The arithmetical work may be very considerably simplified by taking certain multiples of the functions Z, so that all coefficients in the expansions in terms of β are integers.

Let
$$Y = \frac{4(\beta-6)}{\beta H} Z, \tag{15·1}$$

whence we get
$$Y_{11} = -1+\cos 4\theta,$$
$$Y_{22} = \cos 2\theta - \cos 6\theta,$$

and thence the coefficients of $\cos n\theta$ in the expansions for Y_{33}, Y_{44} and p'_{n+1} can be obtained as follows:

n	$16Y_{33}$	$64Y_{44}$	$768\dfrac{\beta-6}{\beta H}p'_{n+1}$
0	$-236+45\beta$	$-904+130\beta$	$-352+61\beta$
2	$304-64\beta$	$1672-230\beta$	$384-63\beta$
4	$-16+20\beta$	$-672+136\beta$	$0+13\beta$
6	-48	$-652-31\beta$	$-224+9\beta$
8	$-4-\beta$	$552-10\beta$	—
10	—	$4+5\beta$	—

The values of Z_{20} have been deduced from these values of p' as in § 10, and the polar and equatorial values of ζ_1 have been computed as follows:

β	Polar values of $\dfrac{\beta-6}{4}\zeta_1$	Equatorial values of $\dfrac{\beta-6}{2}\zeta_1$
10	$H(1-\ 3·7\alpha^2)$	$-H(1-\ 4·1\alpha^2)$
12	$H(1-\ 4·9\alpha^2)$	$-H(1-\ 4·7\alpha^2)$
14	$H(1-\ 7·0\alpha^2)$	$-H(1-\ 4·7\alpha^2)$
16	$H(1-11·0\alpha^2)$	$-H(1-\ 3·3\alpha^2)$
18	$H(1-22·7\alpha^2)$	$-H(1+\ 3·8\alpha^2)$
19	$H(1-45·7\alpha^2)$	$-H(1+20·2\alpha^2)$
...	...	...
21	$H(1+45·5\alpha^2)$	$-H(1-49·9\alpha^2)$

Consider these expressions for $\beta = 18$. When α is zero the polar and equatorial values of ζ_1 have opposite signs; that is, there is only one amphidromic point between pole and equator. It is clear that for quite a small value of α the value of ζ_1 at the pole becomes zero; that is, as α increases, a second amphidromic system develops at the pole and travels towards the equator, and the tides at the pole and equator are then both "indirect". The rapidity of change with α is even more marked with $\beta = 19$.

When, however, we pass through the critical value of β, there is a reversal of the phenomena, for with $\beta = 21$ it is the polar tides which remain unchanged in sign for all small values of α, while the second amphidromic system develops at the equator and travels away from the equator as α increases. Moreover, the tides at the pole and equator are "direct". Thus the tides are completely reversed in sign between $\beta = 19$ and $\beta = 21$, and it is hardly possible to avoid the conclusion that for some intermediate value of β (not necessarily $\beta = 20$) there is resonance.

These results are supported by the investigations of Part V, but they can only be regarded as indications of the character of the changes taking place in the tides as the depths and widths of the ocean change, and there is still a desirability for further elucidation of these changes by analytical methods.

PART V. SOLUTIONS BY USE OF FINITE DIFFERENCES: SEMIDIURNAL TIDES

1. Introduction

The methods used in this memoir are essentially very different in character from those developed in the first four parts, and in one sense they are experimental as the methods of finite differences on such a large scale have never been exploited in connexion with tides in oceans. The great difficulties involved in the use of the common mathematical functions, due to slow convergence of series, even for simplified oceans, appears to set a limit to their use, whereas finite differences can find application, or should be usable, for any types of variations of ocean depths and contours, so that undoubtedly there is a large field of use. In this memoir the methods have been successfully applied to the evaluation of the semidiurnal tide (K_2) in oceans of constant depth up to 90° wide, five cases being evaluated and illustrated. Though it is not expected that the solutions are of great accuracy, yet the variations in the character of the solutions are probably quite genuine, and offer a trustworthy guide to the relation between the dynamical response of the tidal motion and the width of the ocean.

It has been considered unnecessary to print all the tables referred to in the text, and only those tables which have been deemed essential are included herewith, namely, tables 30–34. *The MS. tables, however, have been deposited with the Society and will be available for reference whenever required.*

2. Notation and fundamental equations

The notation and fundamental equations are the same as are used in Part IV, 2 and 3, and it is sufficient to quote equations (3·61), (3·62), (3·71) from Part IV, these being the equations which are subjected to the operations of finite differences. We thus have

$$\nabla^2\zeta_1' + \frac{\partial^2\zeta_1'}{\partial\chi^2} + \eta\frac{\partial\zeta_2'}{\partial\chi} = -\beta H \sin^6\theta\cos 2\chi, \tag{2·11}$$

$$\nabla^2\zeta_2' + \frac{\partial^2\zeta_2'}{\partial\chi^2} - \eta\frac{\partial\zeta_1'}{\partial\chi} = \beta H \sin^6\theta\sin 2\chi, \tag{2·12}$$

with
$$\eta = 2-\sin^2\theta, \tag{2·21}$$
$$\nabla^2 Z = Z''\sin^2\theta - Z'\sin\theta\cos\theta + \beta Z\sin^4\theta, \tag{2·22}$$
where dashes denote differentiation with respect to θ.

The boundary conditions require
$$v_1 = 0, \quad v_2 = 0 \quad \text{on} \quad \chi = \pm\alpha,$$
and from Part IV (3·51) and (3·52), these conditions are satisfied when
$$\frac{\partial \zeta_2'}{\partial \chi} - \sin\theta\cos\theta\frac{\partial \zeta_1'}{\partial \theta} = 0, \tag{2·31}$$
$$\frac{\partial \zeta_1'}{\partial \chi} + \sin\theta\cos\theta\frac{\partial \zeta_2'}{\partial \theta} = 0. \tag{2·32}$$
We shall write
$$x = \sin^2\theta, \tag{2·4}$$
$$P = \zeta_1', \quad Q = \zeta_2', \tag{2·5}$$
and intervals in χ will be denoted by ϵ.

We shall compute P and Q on the meridians for which
$$\chi = (n-\tfrac{1}{2})\,\epsilon, \tag{2·6}$$
where n is an integer.

3. Choice of variables and intervals

It is a matter of some considerable importance to choose variables and intervals with great care, lest the errors associated with the use of finite differences become excessive. Fortunately, the results of Part IV are available for the purpose of testing alternative methods, and experience showed that it was much preferable to choose the prime variable in latitude in the form
$$x = \sin^2\theta,$$
rather than θ, $\sin\theta$, or $\cos\theta$. The conditions near the pole appear to be of equal importance with those near the equator, so that both $\sin\theta$ and $\cos\theta$ are ruled out, as the former gives a preponderance of points near the pole and the latter gives a preponderance of points near the equator. Though this difficulty does not arise with the choice of θ itself, yet the occurrence of such factors as $\sin^6\theta$ in the equations necessitates the consideration of very high orders of differences, whereas when the prime variable is x such a factor only implies the necessity of third differences at most, and in general it has been found that it is much better to choose x rather than θ.

It has been found satisfactory to take intervals of 10° in longitude and 0·1 in x. There is no advantage in taking very small intervals, at least in the methods used in this memoir, for the formulae give sequences of terms with alternating coefficients, and the smaller the interval the greater the rapidity of alternation and the greater the size of the coefficients.

4. Equations in terms of x

Equations (2·21), (2·22), (2·31) and (2·32) become

$$\eta = 2 - x, \tag{4·11}$$

$$\nabla^2 Z = 4x^2(1-x)\frac{\partial^2 Z}{\partial x^2} - 2x^2\frac{\partial Z}{\partial x} + \beta x^2 Z, \tag{4·12}$$

$$V_1 = \frac{\partial \zeta_2'}{\partial \chi} - 2x(1-x)\frac{\partial \zeta_1'}{\partial x}, \tag{4·21}$$

$$V_2 = \frac{\partial \zeta_1'}{\partial \chi} + 2x(1-x)\frac{\partial \zeta_2'}{\partial x}, \tag{4·22}$$

where V_1 and V_2 are proportional to v_1, v_2 respectively.

5. Equations resulting from use of finite differences

We shall take

$$\delta\chi = \epsilon = 0{\cdot}1745329, \tag{5·11}$$

$$\delta x = 0{\cdot}1, \tag{5·12}$$

and we shall denote by $P_{m,n}$, $Q_{m,n}$

the values of ζ_1', ζ_2'

for $x = 0{\cdot}1m$ and $\chi = (n-\frac{1}{2})\,\epsilon$, where m and n are integers.

We shall replace, in general terms,

$$\epsilon\frac{\partial Z_0}{\partial x} \text{ by } \tfrac{1}{2}(Z_1 - Z_{-1}), \tag{5·21}$$

$$\epsilon^2\frac{\partial^2 Z_0}{\partial x^2} \text{ by } (Z_1 - 2Z_0 + Z_{-1}), \tag{5·22}$$

and we then obtain from (2·11) and (2·12) the expressions

$$X = P_{m,n+1} + aQ_{m,n+1}$$
$$= -\beta H\epsilon^2 x^3 \cos 2\chi - P_{m,n-1} + aQ_{m,n-1} - bP_{m-1,n} + cP_{m,n} - dP_{m+1,n}, \tag{5·31}$$

$$Y = Q_{m,n+1} - aP_{m,n+1}$$
$$= \beta H\epsilon^2 x^3 \sin 2\chi - Q_{m,n-1} - aP_{m,n-1} - bQ_{m-1,n} + cQ_{m,n} - dQ_{m+1,n}, \tag{5·32}$$

where

$$a = \tfrac{1}{2}\epsilon\eta, \tag{5·41}$$

$$b = 100\epsilon^2 x^2(4{\cdot}1 - 4x), \tag{5·42}$$

$$c = 2 + 100\epsilon^2(8x^2 - 8x^3 - 0{\cdot}01\beta x^2), \tag{5·43}$$

$$d = 100\epsilon^2 x^2(3{\cdot}9 - 4x). \tag{5·44}$$

It is evident from these equations that if we know P and Q on meridians corresponding to $(n-1)$ and (n) then we have two equations to determine P and Q on meridian $(n+1)$. As it is inconvenient to solve the two equations for each occasion we therefore write

$$P_{m,\,n+1} = fX - gY, \quad Q_{m,\,n+1} = fY + gX, \tag{5·51}$$

where

$$f = \frac{1}{1+a^2}, \quad g = \frac{a}{1+a^2}. \tag{5·52}$$

6. Formulae for points near the equator

Since the variable x, considered non-physically, can proceed beyond unity, it is necessary to consider the points near the equator. On the equator itself we have

$$\nabla^2 Z = \frac{\partial^2 Z}{\partial \theta^2} + \beta Z,$$

and therefore this gives the expression

$$\nabla^2 P_{m,\,n} = \frac{P_{m+1,\,n} - 2P_{m,\,n} + P_{m-1,\,n}}{(\delta\theta)^2} + \beta P_{m,\,n}, \tag{6·1}$$

where m here corresponds to $\theta = 90°$.

Since the corresponding value of θ is such that $\sin^2(\delta\theta) = 0·1$ we have

$$(\delta\theta)^2 = 0·1035234,$$

so that we get

$$c = 2 + \epsilon^2(19·31930 - \beta),$$

$$b = 2 \times 9·65965\epsilon^2,$$

with d suppressed since $P_{m+1,\,n}$ has been written equal to $P_{m-1,\,n}$, because of symmetry about the equator.

Examination of the formulae shows that this modification takes care of all the conditions to be satisfied arising from the fact that the values of the elevations are symmetrical about the equator for semidiurnal tides.

The above formulae have been used in the main computations of this paper, but for the special investigations of the tides in very narrow oceans it was found necessary to use the more accurate formula

$$\nabla^2 P_{10,\,n} = -10P_{8,\,n} + 40P_{9,\,n} - (30-\beta)\,P_{10,\,n}. \tag{6·2}$$

7. Arrangement of computations

It is convenient to arrange the values of P and Q in "cells" under the values of n and against the values of m; though only the functions appearing in the above formulae are given below, it must be understood that all the values of P and Q for any given value of n

are written in the column. The coefficients of those values of P and Q required to give $P_{m,\,n+1}$ and $Q_{m,\,n+1}$ are also written out on corresponding forms, as illustrated below.

Arrangement of functions P and Q		Multipliers for computation of $P_{m,\,n+1}$		Multipliers for computation of $Q_{m,\,n+1}$	
—	$P_{m-1,\,n}$		$-bf$		$-bg$
—	$Q_{m-1\,n}$		bg		$-bf$
$P_{m,\,n-1}$	$P_{m,\,n}$	$-f+ag$	cf	$-af-g$	cg
$Q_{m\,n-1}$	$Q_{m,\,n}$	$af+g$	$-cg$	$-f+ag$	cf
—	$P_{m+1,\,n}$		$-df$		$-dg$
—	$Q_{m+1,\,n}$		dg		$-df$

It should be obvious that when the initial values of P and Q have been written down in the two initial columns for $(n-1)$ and (n) then the strips of paper containing the multipliers can be placed alongside the two columns and the cross-multiplications and summations of the products rapidly effected to give the values of P and Q for $(n+1)$, which are entered in the next column.

Tables 1 and 2 give lists of the multipliers as required above, for $\beta = 10$ and $\beta = 20$. Each line of the tables gives a set of multipliers for a given value of m.

In the case of the particular integral it will be necessary to add to the results of the above process the values of

$$P' = -\beta\epsilon^2x^3H\{f\cos(2n-1)\,\epsilon+g\sin(2n-1)\,\epsilon\}, \tag{7·11}$$

$$Q' = \beta\epsilon^2x^3H\{f\sin(2n-1)\,\epsilon-g\cos(2n-1)\,\epsilon\}, \tag{7·12}$$

and these are tabulated in table 3 for $n = 1$ to 4, and $\beta = 20$. For $\beta = 10$ the values should be divided by 2.

8. The complementary functions

The solution of the problem in view is to be effected by the combination of a number of complementary functions with a particular integral, and there is a very wide choice of such functions. For instance, it would be possible to assume $P = \sin^r\theta$, $Q = 0$ on one meridian, with $Q = \sin^s\theta$ on an adjacent meridian, and by varying r and s sufficiently the problem, no doubt, could be solved. But it has been judged better to leave as unspecified the values of P and Q on the meridians $\chi = \pm\epsilon$ for the 11 special values of $x = 0{\cdot}0, 0{\cdot}1, 0{\cdot}2, \ldots, 1{\cdot}0$. It is clear that the values of P and Q on other meridians will be linear functions of these unspecified quantities.

Since it is known that there must be symmetry in P and asymmetry in Q about the central meridian $\chi = 0$, we thus have standard cases in one of which we take

$$Q = 0,\ P = A_0,\ A_1,\ \ldots,\ A_{10} \text{ on both } \chi = \pm\epsilon,$$

$$P = 0,\ \pm Q = B_1,\ B_2,\ \ldots,\ B_{10} \text{ on } \chi = \pm\epsilon.$$

Since Q must be zero at the pole, we have no value B_0.

It is obvious that the coefficients of each A and B can be worked out independently of the rest, so that a number of "complementary functions" (as they may be called for want of a better term) can be so evaluated. At first sight, when we consider these functions, there may be an appearance of unreality about them, because a function which is zero for all values of x other than a particular value $x = 0{\cdot}1m$ would appear to be so discontinuous as to make it impossible to apply finite-difference formulae. If it is remembered that it is the assemblage of functions which will represent a continuous function, that the justification for the integration by finite-difference formulae depends upon the rapid convergence of the differences of the end-product, and that the "functions" are only coefficients which are being worked out in this manner for the sake of convenience, the difficulty should disappear.

The reason for the choice of meridians $\chi = \pm\epsilon$ is now clear, for the processes of § 6 require the values of P and Q to be known or assumed on two meridians. It is possible to begin with a central meridian but the initial processes are needlessly complicated.

The following table illustrates the early stages of the computations for the complementary function which is the coefficient of A_5, $\beta = 20$. It will be seen that the numerical

EXAMPLE OF COMPUTATIONS OF P AND Q

x		$n=-1$	$n=1$	$n=2$	$n=3$
0·3	P	—	—	—	0·779890
	Q	—	—	—	0·229343
0·4	P	—	—	−1·099556	−8·656653
	Q	—	—	−0·153527	−2·277442
0·5	P	1·00	1·00	3·845115	21·568070
	Q	0·00	0·00	0·372425	5·085188
0·6	P	—	—	−1·836842	−16·410145
	Q	—	—	−0·224412	−3·886281
0·7	P	—	—	—	3·470170
	Q	—	—	—	0·829131

quantities increase rapidly, and alternate in sign. To exhibit the functions in this form, as computed, would take too much space, but tables 5–8 give the coefficients of A_m and B_m in the expansions for P and Q on meridians $n = 2$, 3, and $\beta = 10$, while tables 9–16 give values for $n = 2$, 3, 4, 5 and $\beta = 20$. A comparison with the above standard form will quickly reveal the relationship between the two forms.

9. THE PARTICULAR INTEGRAL

There is a latitude of choice with regard to the Particular Integral. It would be possible, for instance, to take Laplace's solution for an ocean covering the whole earth, but it is simpler in many ways to assume P and Q to be zero on the initial meridians, seeing that special values A and B have been allotted to P and Q on these meridians for the complementary functions.

The procedure for the computations is like that used for the complementary functions, except that it is necessary to add the values of P' and Q' from tables 3 and 4 in order to give P and Q respectively. It should be noted that the values of P' and Q' for $n = 1$ have to be used for the computation of P and Q for $n = 2$, and so on. The results are included in tables 4–16 as the coefficients of H.

10. The boundary equations

The boundary equations resulting from the use of finite differences in (4·21) and (4·22), taking central differences about the bounding meridian as well as along it, yield

$$2{\cdot}8648(Q_{m,n+1}-Q_{m,n-1})-10x(1-x)\,(P_{m+1,n}-P_{m-1,n}), \tag{10·11}$$

$$2{\cdot}8648(P_{m,n+1}-P_{m,n-1})+10x(1-x)\,(Q_{m+1,n}-Q_{m-1,n}), \tag{10·12}$$

as the contributions to V_1 and V_2 respectively.

For the computations, the coefficients are arranged to suit the forms discussed in § 7, and the resulting coefficients of A, B, H in the expansions for V_1 and V_2 are given in tables 17 and 18 for $n = 2$, $\beta = 10$, and in tables 19–24 for $n = 2$, 3 and 4 and $\beta = 20$.

It has been remarked that these formulae use central differences and so cannot be used for the case where the bounding meridian is the one for which $n = 5$. Since the central difference formulae ignore third differences we can obtain suitable formulae, perhaps not quite so accurate as those given above, by assuming third differences to be zero, and thus writing in the standard formulae (10·11) and (10·12)

$$P_{m,6} = 3P_{m,5}-3P_{m,4}+P_{m,3},$$

which yields

$$(2{\cdot}8648Q_{m,3}-11{\cdot}4592Q_{m,4}+8{\cdot}5944Q_{m,5})-10x(1-x)\,(P_{m+1,5}-P_{m-1,5}), \tag{10·21}$$

$$(2{\cdot}8648P_{m,3}-11{\cdot}4592P_{m,4}+8{\cdot}5944P_{m,5})+10x(1-x)\,(Q_{m+1,5}-Q_{m-1,5}), \tag{10·22}$$

as the contributions to V_1 and V_2 respectively.

The resulting expansions are given in tables 25 and 26.

11. The solution of the equations

The 20 simultaneous equations resulting from the application of the boundary conditions have been solved by ordinary methods which call for no special comment. It is, however, necessary to distinguish between the "real accuracy" and "nominal accuracy". The former depends upon the assumptions made in the formulae used, and the computer called upon to solve the resulting equations is not at all concerned with the real accuracy, but he must furnish a solution of the equations which accurately represents those equations. It is essential to maintain the nominal accuracy in this sense because the equations have very large coefficients of alternating sign so that any excessive restriction of the significant figures retained in the computations will inevitably result in large casual errors which will cause much trouble and uncertainty. Hence there has been maintained numerical consistency to as high an order as possible.

It will be noted that there are only 20 equations whereas there are 21 unknowns in A and B to be evaluated in terms of H. The equations have been solved so as to give each unknown in terms of A_0 and H, and in each case the results obtained have been substituted back in the equations for V_1 and V_2, and small adjustments have been made where necessary in order to reduce the residues. The numerical accuracy is very good as the greatest numerical residues for V_1 and V_2 are generally less than $0{\cdot}01H$, and the greatest residue is $0{\cdot}026H$, for $n = 4$, $x = 0{\cdot}9$, $\beta = 20$. The residual values near the pole are extremely small, and this is an important point because the actual velocities vary as $V \operatorname{cosec}^3 \theta$.

It is interesting to compare these residual values of V with the values of V obtained in Laplace's solution for an ocean covering the whole earth. The latter increase from zero at the pole to $5{\cdot}6H$ at the equator for $\beta = 20$ so that the residues we have obtained may be considered as very small indeed.

The solutions giving each A and B in terms of A_0 and H are given in table 27.

12. The determination of A_0

There is only one satisfactory method available for the determination of the quantity A_0 in terms of H, which is to use the theorem of constant volume; that is, we have to make

$$\iint \zeta_1 \sin\theta \, d\theta \, d\chi = 0, \tag{12·1}$$

where the integrals are effected over the whole ocean.

Taking firstly the integral with respect to θ, we have to obtain a formula for

$$\iint_0^{\frac{1}{2}\pi} P \sin\theta \, d\theta, \tag{12·2}$$

where P, as usual, stands for ζ_1'. This can be written as

$$-\int_0^{\prime} P d\mu, \tag{12·31}$$

or, in terms of finite differences,

$$-\Sigma P \,\delta\mu = -\Sigma P \,\delta(\surd(1-x)). \tag{12·32}$$

Taking this formula with $x = 0{\cdot}0, 0{\cdot}1, \ldots, 1{\cdot}0$, we get the formula

$$\tfrac{1}{2}P_0(1-\surd 0{\cdot}9) + \tfrac{1}{2}P_1(1-\surd 0{\cdot}8) + \tfrac{1}{2}P_2(\surd 0{\cdot}9 - \surd 0{\cdot}7) + \ldots. \tag{12·4}$$

An alternative method of considering (12·2) is to integrate it by parts, which yields

$$P_0 + \Sigma\surd(1-x)\,\delta P \tag{12·5}$$

and this gives the formula

$$P_0(1-\sqrt{0\cdot95})+P_1(\sqrt{0\cdot95}-\sqrt{0\cdot85})+\ldots. \tag{12·6}$$

The two formulae have been tested by taking $P = 1$, x, x^2, x^3, and it was found that (12·4) gives results consistently low while (12·6) gives results consistently high. A combination in the ratio of 2 to 1 in favour of (12·6) was adopted, and the factors for P_r are given below.

Factors giving integral with respect to θ

P_0	0·02544	P_4	0·06466	P_8	0·11372
P_1	0·05275	P_5	0·07089	P_9	0·18366
P_2	0·05595	P_6	0·07937	P_{10}	0·20178
P_3	0·05984	P_7	0·09195		

(12·7)

When this formula is applied to the case $P = x$, x^2, x^3 we get the results

0·6658, 0·5332, 0·4577,

the correct values being 0·6667, 0·5333, 0·4571,

so that the formula is a very accurate one.

The later investigations of the tides in very narrow oceans required even more accurate formulae, because the integrals only yielded small quantities. By taking linear combinations of P_m (m even) and solving six simultaneous equations it was possible to obtain formulae accurately representing integrals of powers of x up to x^6, and similar combinations (m odd) gave a formula accurately representing integrals of powers of x up to x^5. In practice the average of these two formulae was used. The coefficients are as follows:

P_0	0·019360	P_1	0·082982
P_2	0·055315	P_3	−0·076471
P_4	0·101010	P_5	0·335689
P_6	0·004810	P_7	−0·190559
P_8	0·202622	P_9	0·348359
P_{10}	0·116883		

(12·8)

With regard to the integral with respect to χ, formulae involving the neglect of fourth differences have been used, derived from

$$\int_0^1 Z\,dz = \tfrac{1}{24}[9Z_0+19Z_1-5Z_2+Z_3], \tag{12·81}$$

$$\int_0^3 Z\,dz = \tfrac{1}{24}[9Z_0+27Z_1+27Z_2+9Z_3]. \tag{12·82}$$

When these are applied to values of Z on the meridians $n = 1, 2, \ldots$ and if note is taken of the fact that there is symmetry about the central meridian, we obtain the following formulae:

$$\text{Integral from } \chi = 0^\circ \text{ to } 15^\circ = \tfrac{1}{24}[27Z_1 + 9Z_2]\,\epsilon, \tag{12·91}$$

$$\text{Integral from } \chi = 0^\circ \text{ to } 25^\circ = \tfrac{1}{24}[23Z_1 + 28Z_2 + 9Z_3]\,\epsilon, \tag{12·92}$$

$$\text{Integral from } \chi = 0^\circ \text{ to } 35^\circ = \tfrac{1}{24}[24Z_1 + 23Z_2 + 28Z_3 + 9Z_4]\,\epsilon, \tag{12·93}$$

$$\text{Integral from } \chi = 0^\circ \text{ to } 45^\circ = \tfrac{1}{24}[24Z_1 + 24Z_2 + 23Z_3 + 28Z_4 + 9Z_5]\,\epsilon, \tag{12·94}$$

where Z_r is any quantity on meridian $n = r$, and ϵ is the interval in χ, here taken equal to 0·17453.

When these formulae are applied to the integral of $\cos 2\chi$ from the central meridian to the meridians for which n is 2, 3, 4, 5 we get the values 0·25004, 0·3831, 0·4700, 0·5002, the correct values being 0·25000, 0·3800, 0·4699, 0·5000 respectively, so that the formulae are amply accurate.

It is a simple matter to apply the formula (12·7) to the coefficients of A, B, and H in tables 5–16 and thence to apply formulae (12·91) to (12·94), including the proper value of ϵ, in order to produce the results given in table 28, for $\beta = 20$ only, which also includes the contributions from $\bar{\zeta}$.

The values of A and B resulting from the solution of the equations and given in table 27 have been multiplied by the factors of table 28, and the sums of the products have yielded equations between A_0 and H, as given below, for $\beta = 20$.

$$\text{Ocean } 30^\circ \text{ wide}: \quad -0·004830A_0 - 0·010453H = 0, \quad A_0 = -2·16420H.$$

$$\text{Ocean } 50^\circ \text{ wide}: \quad -0·067794A_0 + 0·534743H = 0, \quad A_0 = 7·88776H.$$

$$\text{Ocean } 70^\circ \text{ wide}: \quad -0·021725A_0 + 0·106875H = 0, \quad A_0 = 4·91945H.$$

$$\text{Ocean } 90^\circ \text{ wide}: \quad 0·034124A_0 + 0·215190H = 0, \quad A_0 = -6·30604H.$$

In the case of $\beta = 10$ the integrations were performed by the method of (12·4) after the values of P had been computed in terms of A_0 and H, so that no table of factors has been prepared.

13. Computation of ζ_1 and ζ_2

After substituting the values of A_0 in table 27 we obtain the values of A and B in terms of H and it only remains to substitute these in tables 5–16 in order to get the values of P and Q on the standard meridians. These are the values of ζ_1' and ζ_2', and to them must be added the values of $\bar{\zeta}_1$ and $\bar{\zeta}_2$ in order to give the final products, ζ_1 and ζ_2. These results are given in tables 30–34. No values of R and γ have been computed but figures 1–4 give illustrations of the tidal charts ($\beta = 20$ only, see § 14), which have been obtained from the values of ζ_1 and ζ_2 by a graphical process.

TABLE 30. VALUES OF ζ_1 AND ζ_2, OCEAN 30° WIDE

Semidiurnal tide K_2, $\beta = 10$

	$\chi = 5°$		$\chi = 15°$	
x	ζ_1	ζ_2	ζ_1	ζ_2
0·0	0·64	0·00	0·64	0·00
0·1	0·58	−0·04	0·58	−0·12
0·2	0·51	−0·09	0·51	−0·25
0·3	0·43	−0·12	0·42	−0·36
0·4	0·34	−0·16	0·32	−0·45
0·5	0·25	−0·19	0·21	−0·54
0·6	0·15	−0·21	0·08	−0·61
0·7	0·03	−0·24	−0·05	−0·66
0·8	−0·09	−0·26	−0·19	−0·71
0·9	−0·23	−0·27	−0·35	−0·73
1·0	−0·39	−0·28	−0·51	−0·73

TABLE 31. VALUES OF ζ_1 AND ζ_2, OCEAN 30° WIDE

Semidiurnal tide K_2, $\beta = 20$

	$\chi = 5°$		$\chi = 15°$	
x	ζ_1	ζ_2	ζ_1	ζ_2
0·0	−2·16	0·00	−2·16	0·00
0·1	−1·30	0·09	−1·36	0·28
0·2	−0·56	0·12	−0·60	0·37
0·3	0·04	0·09	0·05	0·30
0·4	0·49	0·02	0·55	0·11
0·5	0·80	−0·07	0·87	−0·17
0·6	0·93	−0·18	1·01	−0·48
0·7	0·88	−0·29	0·92	−0·80
0·8	0·61	−0·40	0·59	−1·07
0·9	0·07	−0·48	−0·01	−1·25
1·0	−0·83	−0·53	−0·93	−1·25

TABLE 32. VALUES OF ζ_1 AND ζ_2, OCEAN 50° WIDE

Semidiurnal tide K_2, $\beta = 20$

	$\chi = 5°$		$\chi = 15°$		$\chi = 25°$	
x	ζ_1	ζ_2	ζ_1	ζ_2	ζ_1	ζ_2
0·0	7·89	0·00	7·89	0·00	7·89	0·00
0·1	4·95	−0·46	5·12	−1·37	5·50	−2·25
0·2	2·51	−0·67	2·59	−2·05	2·82	−3·52
0·3	0·54	−0·72	0·44	−2·21	0·28	−3·88
0·4	−1·00	−0·65	−1·29	−1·99	−1·86	−3·51
0·5	−2·07	−0·48	−2·52	−1·49	−3·42	−2·61
0·6	−2·63	−0·25	−3·20	−0·77	−4·28	−1·35
0·7	−2·59	0·05	−3·21	0·10	−4·33	0·10
0·8	−1·79	0·39	−2·43	1·07	−3·44	1·57
0·9	−0·05	0·77	−0·67	2·08	−1·48	2·86
1·0	3·08	1·19	2·44	3·06	1·72	3·65

TABLE 33. VALUES OF ζ_1 AND ζ_2, OCEAN 70° WIDE

Semidiurnal tide K_2, $\beta = 20$

	$\chi = 5°$		$\chi = 15°$		$\chi = 25°$		$\chi = 35°$	
x	ζ_1	ζ_2	ζ_1	ζ_2	ζ_1	ζ_2	ζ_1	ζ_2
0·0	4·92	0·00	4·92	0·00	4·92	0·00	4·92	0·00
0·1	3·00	−0·31	3·12	−0·93	3·39	−1·53	3·75	−2·06
0·2	1·46	−0·48	1·53	−1·45	1·69	−2·44	2·02	−3·46
0·3	0·22	−0·56	0·18	−1·68	0·15	−2·85	0·16	−4·12
0·4	−0·74	−0·57	−0·89	−1·71	−1·17	−2·89	−1·53	−4·14
0·5	−1·38	−0·53	−1·65	−1·58	−2·16	−2·62	−2·87	−3·66
0·6	−1·65	−0·44	−2·04	−1·30	−2·76	−2·11	−3·72	−2·79
0·7	−1·44	−0·30	−1·97	−0·88	−2·86	−1·39	−3·96	−1·66
0·8	−0·65	−0·11	−1·30	−0·32	−2·36	−0·49	−3·47	−0·40
0·9	0·94	0·15	0·13	0·40	−1·08	0·58	−2·12	0·84
1·0	3·68	0·48	2·66	1·31	1·19	1·77	0·33	1·84

TABLE 34. VALUES OF ζ_1 AND ζ_2, OCEAN 90° WIDE

Semidiurnal tide K_2, $\beta = 20$

	$\chi = 5°$		$\chi = 15°$		$\chi = 25°$		$\chi = 35°$		$\chi = 45°$	
x	ζ_1	ζ_2	ζ_1	ζ_2	ζ_1	ζ_2	ζ_1	ζ_2	ζ_1	ζ_2
0·0	−6·31	0·00	−6·31	0·00	−6·31	0·00	−6·31	0·00	−6·31	0·00
0·1	−3·70	0·37	−3·89	1·09	−4·27	1·78	−4·87	2·39	−5·69	2·86
0·2	−1·46	0·57	−1·65	1·71	−2·04	2·90	−2·71	4·16	−3·79	5·48
0·3	0·44	0·65	0·32	1·98	0·11	3·38	−0·26	4·98	−0·82	7·01
0·4	2·00	0·66	1·92	1·99	1·84	3·34	1·92	4·83	2·33	6·81
0·5	3·18	0·62	3·08	1·83	3·04	2·93	3·35	3·96	4·47	5·06
0·6	3·91	0·55	3·73	1·56	3·57	2·32	3·81	2·75	4·96	2·70
0·7	4·08	0·45	3·73	1·21	3·30	1·60	3·21	1·45	3·84	0·68
0·8	3·54	0·33	2·94	0·81	2·09	0·82	1·51	0·17	1·70	−1·50
0·9	2·05	0·18	1·10	0·36	−0·30	0·07	−1·54	−0·68	−2·28	−1·52
1·0	−0·88	0·01	−2·30	−0·16	−4·46	−0·73	−6·19	−1·48	−6·98	−1·81

14. COMPARISON WITH SERIES METHOD OF PART IV

The solution obtained in this present part for $\beta = 10$ is practically the same as that obtained in Part IV by the series method, and the tidal charts are practically indistinguishable from one another, so that it is not necessary to illustrate for $\beta = 10$ the results of this Part. At the pole the values of ζ_1' are respectively $0{\cdot}69H$ and $0{\cdot}64H$ from Parts IV and V. On the bounding meridians at the equator the values of ζ_1' are $-1{\cdot}35H$ and $-1{\cdot}38H$, while the values of ζ_2' are $-0{\cdot}23H$ and $-0{\cdot}19H$. The amphidromic points are situated on the central meridian at $\theta = 57°{\cdot}8$ (Part IV) and $\theta = 59°{\cdot}1$ (Part V). The agreement between the two methods is considered to be quite satisfactory.

15. VERY NARROW OCEANS

The investigations made in Part IV relative to a very narrow ocean have been continued by the methods of finite differences for oceans 10° wide ($\alpha = 5°$) and $\beta = 18$, 20,

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and 22. The equations for V_1 and V_2 are given in tables 35 and 36 for $\beta = 20$, and the corresponding tables for $\beta = 18$ and 22 are very simply derived as instructed at the feet of these tables. The resulting values of A and B in terms of A_0 and H are given in table 37, from which, after including $\bar{\zeta}_1$, we obtain by the application of (12·8) the following equations:

$$\beta = 18, \quad -0{\cdot}0442A_0 - 0{\cdot}0083H = 0, \quad \text{whence } A_0 = -0{\cdot}19H, \tag{15·1}$$

$$\beta = 20, \quad -0{\cdot}0171A_0 - 0{\cdot}0169H = 0, \quad \text{whence } A_0 = -0{\cdot}99H, \tag{15·2}$$

$$\beta = 22, \quad 0{\cdot}0018A_0 - 0{\cdot}0216H = 0, \quad \text{whence } A_0 = 11{\cdot}7H. \tag{15·3}$$

From the resulting values of ζ_1 we deduce the positions of the amphidromic points on the central meridian as follows:

$$\beta = 18, \quad \theta = 16^\circ \text{ and } 65^\circ,$$
$$\beta = 20, \quad \theta = 27^\circ \text{ and } 69^\circ,$$
$$\beta = 22, \quad \theta = 32^\circ \text{ and } 72^\circ.$$

These results can be considered in relation to the results of the investigations of Part IV, § 15, so long as we remember that we cannot expect very accurate numerical consistency on account of the errors of approximation inherent in each method. In Part IV we deduced that for any given value of β approaching the critical value $\beta = 20$, there is a definite value of width of ocean marking the division between the tidal regime with only one amphidromic system, and that with two amphidromic systems, between pole and equator, and for this particular value of α the tides vanish at the pole. This deduction is strongly confirmed by the above results, since it is evident that for the case there considered ($\alpha = 5^\circ$) the value of A_0 is zero when β is approximately equal to 17.

Another general deduction obtained in Part IV was to the effect that resonance occurs for values of β somewhat greater than the critical value ($\beta = 20$), this taking place for a value of α appropriate to the depth of the ocean. This conclusion is also vindicated by the above results, since it is clear that resonance takes place when $\beta = 21{\cdot}8$, approximately.

For values of β near the critical value, it seems quite certain that though the tidal regime may be of the single amphidromic type for an ocean of infinitesimal width, yet there are very rapid changes as the width becomes appreciable, with a rapid transition to the dual system of amphidromic points between pole and equator.

16. Investigations by Goldsbrough and Colborne

The lunar semidiurnal tide (M_2) has been determined by Goldsbrough and Colborne (1929) for an ocean 60° wide and constant depth corresponding to $\beta = 22{\cdot}9$. The method used is quite distinct from that used in any of this present series of memoirs and it is desirable to discuss the results now obtained in relation to those obtained by Goldsbrough and Colborne.

Goldsbrough and Colborne's results indicate approximate resonance, with a maximum amplitude of $37H$, and a characteristic feature is that the amplitude of tide at the pole is comparatively very small indeed, being only $0{\cdot}5H$.

It might be expected that where the tides are small over a large region of the ocean, the cotidal systems may become very complicated, and the chart of cotidal lines as given by Goldsbrough and Colborne shows a very complex system of four amphidromic points between pole and equator, in our notation, as follows:

one at $\theta = 61^\circ{\cdot}5$ on the central meridian,
one at $\theta = 44^\circ$ on the central meridian,
two at $\theta = 34^\circ$ on the meridians $\chi = \pm 18^\circ$.

The closeness of these amphidromic systems raises difficulties of interpretation, so much so that the question needed to be considered as to whether the systems were real or simply due to inaccuracies of the solution. Some weight was given to this doubt by the fact that the amplitudes of tide in the region are very small and only two or three times the known errors of the solution (judging from the fact that the solution gives amplitudes of $37{\cdot}6H$ and $36{\cdot}6H$ at the boundaries on the equator, and as these should be equal, the possible error appears to be $0{\cdot}5H$). Further, one would naturally expect the null lines on ζ_1 and ζ_2 to be more regularly spaced than the solution indicated.

Professor Goldsbrough kindly supplied the values of the elevations computed for values of μ at intervals of $0{\cdot}1$, and the values of χ at intervals of 10°. A scrutiny of these immediately yielded the result that an amphidromic system near the pole had been overlooked, at $\theta = 6^\circ$, $\chi = 0$, but otherwise the numerical values appear to offer no support to the possibility that the complex systems were due largely to errors of approximation.

Further remarks on this work will be made in relation to the discussion of the results of this memoir, and, to facilitate the comparisons, Goldsbrough and Colborne's chart has been redrawn with the new amphidromic system inserted near the pole, and co-range lines have been added. The chart has been figured in conformity with the notation of this memoir.

17. Discussion of results

If we consider the results as depicted in the charts we see at once that there is a very marked stability of tidal regime as the width of the ocean increases. There are two amphidromic systems around points on the central meridian as follows:

$\beta = 20$, ocean 10° wide, $\theta = 27^\circ$ and 69°,
$\beta = 20$, ocean 30° wide, $\theta = 31^\circ$ and 71°,
$\beta = 20$, ocean 50° wide, $\theta = 35^\circ$ and 72°,
$\beta = 20$, ocean 70° wide, $\theta = 35^\circ$ and 67°,
$\beta = 20$, ocean 90° wide, $\theta = 31^\circ$ and 81°.

Very close agreement between the positions of the amphidromic points thus occurs for all widths of oceans. This may be taken as valuable testimony to the accuracy of the results for the very narrow ocean; it was a possibility to be considered that as the equations between A_0 and H for the narrow ocean involved relatively small quantities, the equations might have been affected by the errors of the finite-difference methods.

Another highly confirmatory fact is that the zero values for $P_4(\cos\theta)$ occur at $\theta = 30{\cdot}6$ and $70{\cdot}1$. It is evident, therefore, that this function is dominant for $\beta = 20$, though in the theories discussed in Part IV this function is only associated with a free motion so long as the ocean is extremely narrow. In view of the conclusions of § 15, it is probable that this dominance comes into effect as soon as the width becomes appreciable, but there is need for more precise analytical investigation before this matter can be satisfactorily explained.

Similarly, it is interesting to notice that for an ocean 50° wide the polar tides are direct and they remain so for the ocean 70° wide, but they become indirect again for the ocean 90° wide. These effects can be investigated by referring to the equations relating A_0 and H as given at the end of § 12, from which it will be seen that, assuming continuity, the coefficients of A_0 are negative for the oceans of width 30°, 50°, 70° and become positive for the ocean of width 90°, whereas the coefficients of H are positive for the last three cases and negative for the first one. We conclude:

(*a*) that for an ocean rather more than 30° wide the coefficient of H becomes zero and therefore we get $A_0 = 0$;

(*b*) that for an ocean approximately 78° wide the coefficient of A_0 must become infinite, indicating a resonant case.

We see therefore that there are two ways in which the polar tides may change from "indirect tides" to "direct tides" and vice versa. (It may be noted in passing that the coefficients of H appear to reach a minimum between $2\alpha = 70°$ to $90°$, and if this minimum happens to be less than zero a very interesting and complicated sequence of tidal charts can be contemplated, as both the above cases would exist in rapid sequence.)

Goldsbrough and Colborne's solution appears to come under case (*a*) and this possibility removes a difficulty which had existed in comparing their solution with an expression later obtained by Goldsbrough (1933, p. 250, case ii) for the free oscillations of an ocean 60° wide, $\beta = 22{\cdot}54$, and a semidiurnal tide of rather smaller speed than that of M_2. Goldsbrough's results, even allowing for the smaller terms he evaluates, show that the tide is represented by an expression with a predominant term proportional to $P_4(\cos\theta)$. He expresses the opinion that more terms might reproduce the four amphidromic points of the earlier paper (five as now amended) whereas the approximate results do not give more than the two amphidromic points referred to above for the very narrow ocean. He does not remark, however, on the discrepancies as regards the distribution of elevation. In the 1929 paper, as we have stated, the polar tide was negligibly small, whereas the 1933 paper would indicate a polar tide over twice as great as the equatorial tide.

The above discussion reveals that the case chosen by Goldsbrough and Colborne happens to be rather abnormal.

No special comment needs to be made on figure 2, for an ocean 50° wide, as the amphidromic systems are of the normal type, but the charts given in figures 3 and 4 are very interesting in connexion with the amphidromic systems in low latitudes. The former chart shows an amphidromic system in which the cotidal lines, instead of radiating at approximately equal intervals, are osculatory. This type of amphidromic system has been revealed once previously, for the diurnal tides in a hemispherical ocean on a non-rotating earth (Proudman and Doodson 1927, figure 6).

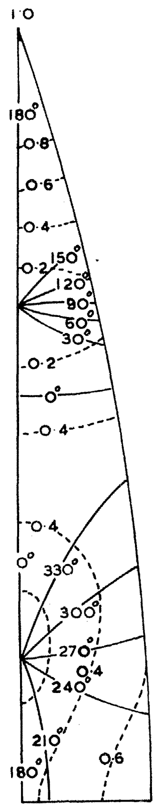

FIGURE 1. Semidiurnal tide K_2 in ocean 30° wide. $\beta = 20$, depth = 2·75 miles, factor = 2·2H.

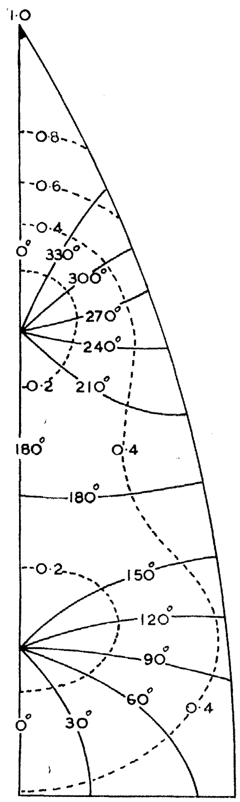

FIGURE 2. Semidiurnal tide K_2 in ocean 50° wide. $\beta = 20$, depth = 2·75 miles, factor = 7·9H.

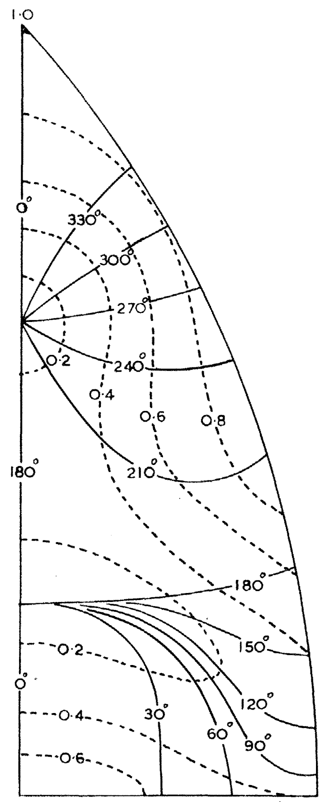

FIGURE 3. Semidiurnal tide K_2 in ocean 70° wide. $\beta = 20$, depth = 2·75 miles, factor = 4·9H.

The explanation of this system and the system shown in figure 4 for the ocean 90° wide, where many of the lines are very close to one another, is found by studying the null lines of ζ_1 and ζ_2. These null lines can be considered as nodes in oscillations, and the amphidromic points occur, of course, at the intersections of the two sets of lines. The null line for ζ_1 is represented by the cotidal line $\gamma = 90°$ or $270°$ and it is evident from figures 1–4 that this line is approximately stationary, though it has moved towards the equator in the last figure. One of the null lines for ζ_2, represented by the cotidal line $\gamma = 0°$ or $180°$,

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moves steadily to the equator. In figure 3 it happens to be tangential to the null line ($\gamma = 90°, 270°$) for ζ_1, giving the osculatory case referred to and in figure 4 it has passed through the null line for ζ_1, but as it remains near to it the cotidal lines remain crowded together near the central meridian. But only the central part of the null line $\gamma = 0°, 180°$, has crossed the null line $\gamma = 90°, 270°$, so that at the intersections amphidromic points develop in longitudes $\chi = 30°$, approximately.

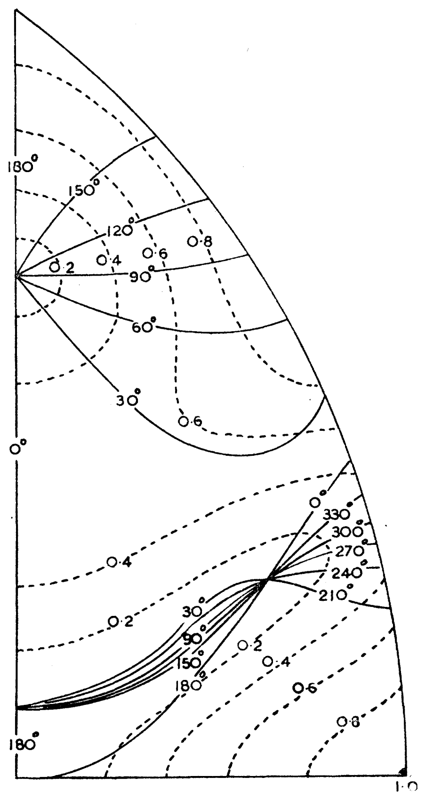

FIGURE 4. Semidiurnal tide K_2 in ocean 90° wide. $\beta = 20$, depth $= 2{\cdot}75$ miles, factor $= 6{\cdot}3H$.

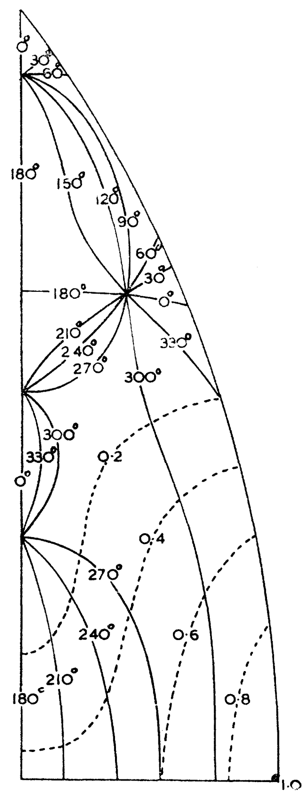

FIGURE 5. Semidiurnal tide M_2 in ocean 60° wide. $\beta = 22{\cdot}9$, factor $= 37H$ (after Goldsbrough and Colbourne).

It is evident also that a similar explanation accounts for the complex amphidromic system obtained by Goldsbrough and Colborne, and the positions of their null lines can readily be traced from figure 5.

No attempt will here be made to discuss further explanations in general terms as it is hoped to consider in a later memoir the whole of the results of these memoirs together, in order to yield a comprehensive general dynamical explanation.

The author is again greatly indebted to Miss M. M. Gill for valuable assistance in connexion with the numerical work.

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